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R-matrix and Clebsch–Gordan coefficients for semiperiodic representation of $SU_q(2)$

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Abstract. By using the method proposed by Gomez and Sierra, I give the explicit expression of the intertwiner for semiperiodic representations of $SU_q(2)$ when $q^N=1$. The quantum Clebsch–Gordan coefficient for this case was also discussed.

1. Introduction

Recently, much more attention have been paid on the quantum group which is non-commutative and non-cocommutative Hopf algebra developed in statistical models. The physical applications comprise lattice statistical models [1–5] and conformal field theory [6–9]. There are deep relations to other theory such as braid group which has been related to fractional quantum Hall effect [10].

One of the simplest examples of quantum groups is $SU_q(2)$, the deformation of $SU(2)$, which is the hidden symmetry of minimal and $SU(2)$ Wess–Zumio–Witten models in conformal field theory [6, 9] and of Heisenberg model with non-trivial boundary condition in statistical model [3]. In conformal field theory, the q-analogue of $6j$ symbol is related to the crossing and braiding properties of conformal blocks [6, 9]. It is well known that $SU_q(2)$ with generic q has similar properties as $SU(2)$. The Racah and Clebsch–Gordan coefficients have been explicitly given [11, 12].

However, the physical interest one is the case q being a root of unit. The symmetries of most physical models are the quantum groups in this case. Mathematically, things become complicated. For $SU_q(2)$, the representations can be classified into four kinds [13]: (a) irreducible or regular representation which can be characterized by spin j as classical $SU(2)$; (b) semiperiodic representation which has a non-zero parameter; (c) periodic or cyclic representation; (d) indecomposable representation. Forthermore, there is no expression of R-matrix, but only for pieces of R matrices acting on a particular type of representation. It seems desirable to find the R-matrix satisfying the Yang–Baxter equation and the Clebsch–Gordan coefficients for semiperiodic representation of $SU_q(2)$.

The R-matrix in this case can be obtained by solving an operator iterating equation which is derived from the axiom of quasi-triangular Hopf algebra. Gomez and Sierra [14, 15] proved the existence of the R-matrix satisfying the Yang–Baxter equation under the condition that the parameters characterizing two semiperiodic representations lie on an algebraic curve. This R-matrix can be used to construct a statistical model. In this paper, we use the method proposed by Gomez and Sierra to get the intertwiner

R and the Clebsch–Gordan coefficients for the semiperiodic representation of $SU_q(2)$. The programme is reviewing the algebra $SU_2(2)$ in section 2 and calculating the intertwiner R and giving the Clebsch–Gordan coefficients in sections 3 and 4 respectively. In the following we will think q as a root of unit ($q^N=1$, N odd).

2. $SU_q(2)$ algebra

$SU_q(2)$ algebra is a Hopf algebra with generators E, F and K which satisfy

$$\begin{aligned} &EF-q^2FE=1-K^2\\ &KE=q^{-2}EK\\ &KF=q^2FK \end{aligned} \tag{1}$$

the comultiplication is

$$\begin{aligned} &\Delta E=E\otimes 1+K\otimes E\\ &\Delta F=F\otimes K^{-1}+1\otimes F\\ &\Delta K=K\otimes K \end{aligned} \tag{2}$$

antipode γ and co-unit ε are

$$\begin{aligned} &\gamma(E)=-K^{-1}E \qquad \gamma(F)=-K^{-1}F \qquad \gamma(K)=K^{-1}\\ &\varepsilon(E)=\varepsilon(F)=0 \qquad \varepsilon(K)=1. \end{aligned} \tag{3}$$

For case $q^N=1$ (N odd), the central Hopf subalgebra is generated by $x=E^N$, $y=F^N$, $z=K^N$ and casmir $C=EF+[H+1]/2(K=q^H)$. The representation of $SU_q(2)$ can be classified according to the values of x, y, z. Our interested one is so-called semiperiodic representation ($x=0$, $y, z\neq 0$), which is an irreducible representation of dimension N. In this case, E is a nilpotent whereas F is an injective. So, the representation can be denoted by $\xi=(y,\lambda)$. It has a highest weight vector v_0 such that $Ev_0=0$ and no lowest weight vector. The semiperiodic representation is given by the basis $(v_0,\ldots,v_{N-1})$ [15, 16] and

$$\begin{aligned} &Fv_p=k^{1/N}(1-\lambda q^p)v_{p+1}\\ &Ev_p=k^{1/N}[p](1+\lambda q^{p-1})v_{p-1}\\ &Kv_p=\lambda q^{2p}v_p \end{aligned} \tag{4}$$

where $k=1/(1-\lambda^N)$ and $[p]$ is the q-number

$$[p]=\frac{1-q^{2p}}{1-q^2}. \tag{5}$$

For the convenience, we will use the notes introduced by Gomez and Sierra, which is

$$(\lambda)_{p_1,p_2}=\begin{cases} \prod_{r=p_1}^{p_2-1}(1-\lambda q^r) & p_1<p_2\\ 1 & p_1=p_2\\ \frac{[p_1]!}{[p_2]!}\prod_{r=p_2}^{p_1-1}(1+\lambda q^2) & p_1>p_2. \end{cases} \tag{6}$$

In terms of this equation (4) can be written as

$$\begin{aligned} &F^ne_p=(\lambda)_{p,p+n}e_{p+n}\\ &E^ne_p=(\lambda)_{p,p-n}e_{p-n}. \end{aligned} \tag{7}$$

That quantum group $SU_q(2)$ is a quasitriangular Hopf algebra means the existence of a map R

$$R: SU_q(2)\otimes SU_q(2) \to SU_q(2)\otimes SU_q(2) \tag{8}$$

which must satisfy

$$R\Delta(a) = \Delta^l(a)R \qquad (a \in SU_q(2)) \tag{9}$$

$$\begin{aligned}(id\otimes\Delta)R &= R_{13}R_{12}\\ (\Delta\otimes id)R &= R_{13}R_{23}\end{aligned} \tag{10}$$

where $\Delta^l(a) = \sigma\circ\Delta(a)$. Equation (10) means that R satisfies the Yang–Baxter equation:

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}. \tag{11}$$

For generic q, it has been shown that the universal R-matrix can be written as

$$R = q^{(H\otimes H)/4}\sum_{n\geq 0}\frac{q^{n(n+1)/2}}{[n]!}E^n\otimes q^{-nH/2}F^n. \tag{12}$$

However, there is no expression for an universal R-matrix, but only for pieces of R matrices acting on particular type of representations in case q being root of unit. We will follow the method proposed in [15] to find the R-matrix for $SU_q(2)$. Now we only consider the intertwiner between two semiperiodic representations which are characterized by ξ_1 and ξ_2 respectively. Assuming that the intertwiner satisfies the following condition

$$1\colon R(\xi,\xi) = I_{N\otimes N} \qquad \text{normalization} \tag{13}$$

$$2\colon R(\xi_1,\xi_2)R(\xi_2,\xi_1) = I_{N\otimes N} \qquad \text{unitarity} \tag{14}$$

$$3\colon R(\xi_1,\xi_2) = PR(\xi_2,\xi_1)P \qquad \text{reflection symmetry.} \tag{15}$$

Generally, equations (10) and (14) uniquely determine an intertwiner R up to a constant which can be fixed by proper normalization. For the convenience, we may normalize it according to the action of R on the highest weight vector state $e_0\otimes e_0$ to be a permutation:

$$R(\xi_1,\xi_2)e_0(\xi_1)\otimes e_0(\xi_2) = P(e_0(\xi_2)\otimes e_0(\xi_1)). \tag{16}$$

From equations (9) and (15), one can get an important relation

$$R(\xi_1,\xi_2)P\Delta_{\xi_1\xi_2}(a)P = P\Delta_{\xi_2\xi_1}(a)PR(\xi_1,\xi_2) \tag{17}$$

here $a\in SU_q(2)$. With the help of F^n acting on the highest weight vector, one can get all vectors in the representation. So, we take the a in equations (9) and (17) as F from which one can find the iterating relation

$$R(\xi_1,\xi_2)(F_1\otimes 1) = [AR(\xi_1,\xi_2) - BR(\xi_1,\xi_2)(K_1\otimes 1)]\frac{1}{1-K_1\otimes K_2} \tag{18}$$

$$R(\xi_1,\xi_2)(1\otimes F_2) = [BR(\xi_1,\xi_2) - AR(\xi_1,\xi_2)(1\otimes K_2)]\frac{1}{1-K_1\otimes K_2} \tag{19}$$

where A, B are given by

$$\begin{aligned}A &= \Delta_{\xi_2\xi_1}(F) = F_2\otimes 1 = K_2\otimes F_1\\ B &= (\sigma\circ\Delta)_{\xi_2\xi_1} = 1\otimes F_1 + F_2\otimes K_1.\end{aligned} \tag{20}$$

From equations (18) and (19), one find

$$R(\xi_1, \xi_2)(F_1^{r_1} \otimes F_2^{r_2})$$
$$= \left(\sum_{s_1=0}^{r_1} (-1)^{s_1} q^{s_1(s_1-1)} \begin{bmatrix} r_1 \\ s_1 \end{bmatrix} A^{r_1-s_1} B^{s_1} \right)$$
$$\times \left(\sum_{s_2=0}^{r_2} (-1)^{s_2} q^{s_2(s_2-1)} \begin{bmatrix} r_2 \\ s_2 \end{bmatrix} B^{r_2-s_2} A^{s_2} \right)$$
$$\times R(\xi_1, \xi_2)(K_1^{s_1} \otimes K_2^{s_2}) \frac{1}{\prod_{l=0}^{r_1+r_2-1} (1-q^{2l} K_1 \otimes K_2)}. \tag{21}$$

Applying the above equation to the highest weight vector $e_0(\xi_1) \otimes e_0(\xi_2)$, one can obtain the explicit form of $R(\xi_1, \xi_2)$

$$R(\xi_1, \xi_2)(F_1^{r_1} e_0(\xi_1) \otimes F_2^{r_2} e_0(\xi_2))$$
$$= \frac{1}{\prod_{l=0}^{r_1+r_2-1} (1-q^{2l} \lambda_1 \lambda_2)}$$
$$\times \prod_{l_1=0}^{r_1-1} (A - \lambda_1 q^{2l_1} B) \prod_{l_2=0}^{r_2-1} (B - \lambda_2 q^{2l_2} A) e_0(\xi_2) \otimes e_0(\xi_1). \tag{22}$$

Expanding the left-hand side of equation (22), one can find the matrix elements $R_{r_1 l_2}^{r_2' r_1'}$ which will be given the next section.

3. The explicit expression of intertwiner

In the previous section, we give the operator form of intertwiner R. In order to find the explicit expression of R, we must expand the left-hand side of equation (22) in term of $F_2^n \otimes F_1^m$. Define two operators as

$$M_i = A - \lambda_q^{2i} B = F_2 \otimes (1 - \lambda_1 q^{2i} K_1) + (K_2 - \lambda_1 q^{2i}) \otimes F_1 \tag{23}$$
$$N_i = B - \lambda_q^{2i} A = F_2 \otimes (K_1 - \lambda_2 q^{2i}) + (1 - \lambda_2 q^{2i} K_2) \otimes F_1 \tag{24}$$

acting on the base $e_s(\xi_2) \otimes e_t(\xi_1)$, which give

$$M_i e_s(\xi_2) \otimes e_t(\xi_1)$$
$$= (\lambda_2)_{s,s+1} (1 - \lambda_1^2 q^{2(i+t)}) e_{s+1}(\xi_2) \otimes e_t(\xi_1)$$
$$+ (\lambda_1)_{t,t+1} (\lambda_2 q^{2s} - \lambda_1 q^{2i}) e_s(\xi_2) \otimes e_{t+1}$$
$$= a_i(s, t) e_{s+1} \otimes e_t + b_i(s, t) e_s \otimes e_{t+1} \tag{25}$$

$$N_i e_s(\xi_2) \otimes e_t(\xi_1)$$
$$= (\lambda_2)_{s,s+1} (\lambda_1 q^{2t} - \lambda_2 q^{2i}) e_{s+1}(\xi_2) \otimes e_t(\xi_1)$$
$$+ (\lambda_1)_{t,t+1} (1 - \lambda_2^2 q^{2(i+s)}) e_s(\xi_2) \otimes e_{t+1}$$
$$= c_i(s, t) e_{s+1} \otimes e_t + d_i(s, t) e_s \otimes e_{t+1} \tag{26}$$

and also define two functions ψ and $\tilde{\psi}$ as

$$\prod_{i=0}^{p} M_i e_s \otimes e_t = \sum_{j=0}^{p+1} \psi_j^{p+1}(s, t, \lambda_1, \lambda_2) e_{s+p+1-j} \otimes e_{t+j} \tag{27}$$

$$\prod_{i=0}^{p} N_i e_s \otimes e_t = \sum_{j=0}^{p+1} \tilde{\psi}_j^{p+1}(s, t, \lambda_1, \lambda_2) e_{s+j} \otimes e_{t+p+1-j}. \tag{28}$$

Substituting equations (27) and (28) into equation (22), one can find

$$\begin{aligned} R(\xi_1, \xi_2)&(F_1^{r_1} e_0(\xi_1) \otimes F_2^{r_2} e_0(\xi_2)) \\ &= \sum_{\mu=0}^{r_1+r_2} \left\{ \sum_{l=0}^{r_2} \psi_{r_1+l-\mu}^{r_1}(s, t, \lambda_1, \lambda_2) \tilde{\psi}_l^{r_2}(s+r_1+l-\mu, t+\mu-l, \lambda_1, \lambda_2) \right. \\ &\quad \left. \times \prod_{l=0}^{r_1+r_2-1} \frac{1}{1-q^{2l}\lambda_1\lambda_2} e_{s+\mu}(\xi_2) \otimes e_{t+r_1+r_2-\mu}(\xi_1) \right\}. \end{aligned} \tag{29}$$

In order to find the elements of intertwiner R, one must get the functions defined in (27) and (28). It is convenient to graphic representation. The equations (25) and (26) can be depicted in figure 1 with factors on each edge as

$$\begin{aligned} a_i(s, t) &= (\lambda_2)_{s,s+1}(1-\lambda_1^2 q^{2(i+t)}) \\ b_i(s, t) &= (\lambda_1)_{t,t+1}(\lambda_2 q^{2s} - \lambda_1 q^{2i}) \\ c_i(s, t) &= (\lambda_2)_{s,s+1}(\lambda_1 q^{2t} - \lambda_2 q^{2i}) \\ d_i(s, t) &= (\lambda_1)_{t,t+1}(1-\lambda_2^2 q^{2(i+s)}). \end{aligned}$$

Then equations (27) and (28) can be depicted in figure 2. Every vertex stands for a vector state $e_n \otimes e_m$ (n, m), and every directed edge together with its factor represents the linear relation in (27) and (28). All vertices at the same horizontal line has the same $n+m$, and the subindex $n(m)$ decreases (increases) one by one from left to right. A path between two vertices is admissible if all edges in it are along the path. From the top vertex to someone in the bottom, there are many admissible paths. Hence, we have

$$\psi_j^{p+1} = \sum \text{admissible path}. \tag{30}$$

With the help of this hint, one can get the expression of ψ_j^{p+1} as

$$\begin{aligned} \psi_j^{p+1}&(s, t, \lambda_1, \lambda_2) \\ &= (\lambda_2)_{s,s+p+1-j}(\lambda_1)_{t,t+j} \prod_{l=0}^{j-1} (\lambda_2 q^{2s} - \lambda_1 q^{2l}) \\ &\quad \times \prod_{l=t+j}^{t+p} (1-\lambda_1^2 q^{2l}) \frac{[p+1]!}{[j]![p+1-j]!}. \end{aligned} \tag{31}$$

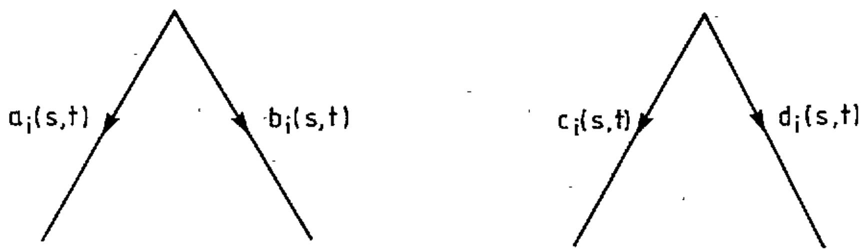

Figure 1. Representation of the relations (25) and (26).

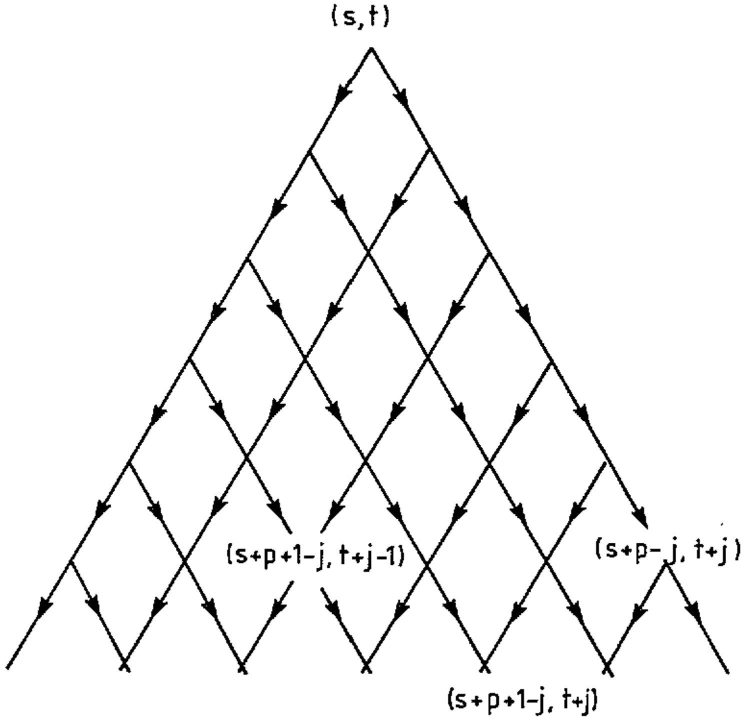

Figure 2. Graph used for the calculation of M_l and N_l.

The proof of the above equation is direct calculation. From figure 2, we know that the admissible paths from $(s+p-j, t+j)$ and $(s+p+1-j, t+j-1)$ are uniquely determined by the edges with factor a_p and b_p respectively. So we have

$$
\begin{aligned}
&\psi_j^{p+1}(s,t,\lambda_1,\lambda_2)\\
&\quad=\psi_j^p(s,t,\lambda_1,\lambda_2)a_p+\psi_{j-1}^p(s,t,\lambda_1,\lambda_2)b_p\\
&\quad=(\lambda_2)_{s,s+p-j}(\lambda_1)_{t,t+j}\prod_{l=0}^{j-1}(\lambda_2q^{2s}-\lambda_1q^{2l})\prod_{l=t+j}^{t+p-1}(1-\lambda_1^2q^{2l})\frac{[p]!}{[j]![p-j]!}\\
&\qquad\times(\lambda_2)_{s+p-j,s+p+1-j}(1-\lambda_1^2q^{2(p+t+j)})\\
&\qquad+(\lambda_2)_{s,s+p+1-j}(\lambda_1)_{t,t+j-1}\prod_{l=0}^{j-2}(\lambda_2q^{2s}-\lambda_1q^{2l})\\
&\qquad\times\prod_{l=t+j-1}^{t+p-1}(1-\lambda_1^2q^{2l})\frac{[p+1]!}{[j]![p+1-j]!}(\lambda_1)_{t+j-1,t+j}(\lambda_2q^{2(s+p+1-j)}-\lambda_1q^{2p})\\
&\quad=(\lambda_2)_{s,s+p+1-j}(\lambda_1)_{t,t+j}\prod_{l=0}^{j-1}(\lambda_2q^{2s}-\lambda_1q^{2l})\prod_{t+j}^{t+p-1}(1-\lambda_1^2q^{2l})\frac{[p]!}{[j]![p+1-j]!}\\
&\qquad\times\{[p+1-j](1-q^{2(p+t+j)}\lambda_1^2)+[j]q^{2(p-j+1)}(1-q^{2(t+j-1)}\lambda_1^2)\}
\end{aligned}
$$

it is easy to show that the right-hand side of the above equation coincides with one of (30). The function $\tilde{\psi}_j^{p+1}$ can be obtained by using the similar way. We do not repeat the procedure and only give the final result

$$
\tilde{\psi}_j^{p+1}(s,t,\lambda_1,\lambda_2)=\psi_j^{p+1}(\lambda_1\leftrightarrow\lambda_2,\ s\leftrightarrow t). \tag{32}
$$

In the above, $(x\leftrightarrow y)$ represents the permutation of x and y. Substituting the equations (31) and (32) into (22) and taking care of the action of F on e_r, we get the elements of intertwiner R

$$\begin{aligned}R_{r_1r_2}^{\mu r_1+r_2-\mu}=&\frac{1}{(\lambda_1)_{0,r_1}(\lambda_2)_{0,r_2}\prod_{l=0}^{r_1+r_2-1}(1-q^{2l}\lambda_1\lambda_2)}\\&\times\sum_{l=0}^{\mu}\psi_{r_1+l-\mu}^{r_1}(0,0,\lambda_1,\lambda_2)\psi_l^{r_2}(r_1+l-\mu,\mu-l,\lambda_2,\lambda_1)\\=&\frac{(\lambda_2)_{0,\mu}(\lambda_1)_{0,r_1+r_2-\mu}}{(\lambda_1)_{0,r_1}(\lambda_2)_{0,r_2}}\prod_{l=0}^{r_1+r_2-1}(1-q^{2l}\lambda_1\lambda_2)\\&\times\sum_{l=0}^{r_1}\left\{\prod_{j=0}^{r_1+l-\mu-1}(\lambda_2-\lambda_1q^{2j})\prod_{j=0}^{l-1}(\lambda_1q^{2(r_1+l-\mu)}-\lambda_2q^{2j})\right.\\&\times\prod_{j=r_1+l-\mu}^{r_1}(1-\lambda_1^2q^{2j})\prod_{j=\mu}^{r_2+\mu-l-1}(1-\lambda_2^2q^{2j})\\&\left.\times\frac{[r_1]![r_2]!}{[r_1+l-\mu]![\mu-l]![r_2-l]![l]!}\right\}.\end{aligned}\tag{33}$$

Now, we consider some special cases. The simplest two elements are

$$R_{r_1r_2}^{N0}=\frac{\prod_{l=0}^{r_1-1}(1-\lambda_1^2q^{2l})\prod_{l=0}^{r_2-1}(\lambda_1-\lambda_2q^{2l})}{\prod_{l=0}^{r_1+r_2-1}(1-\lambda_1\lambda_2q^{2l})}\frac{(\lambda_2)_{0,N}}{(\lambda_1)_{0,r_1}(\lambda_2)_{0,r_2}}\qquad(r_1+r_2=N)\tag{34}$$

$$R_{r0}^{r-nn}=\frac{(\lambda_1)_{r,n}(\lambda_2)_{0,r-n}}{[r-n]!}\frac{\prod_{l=0}^{n-1}(\lambda_2-\lambda_1q^{2l})}{\prod_{l=0}^{r-1}(1-\lambda_1\lambda_2q^{2l})}.\tag{35}$$

The element R_{r0}^{r-nn} appeared in [15]. When N approaches to ∞, equation (33) gives a nontrivial limit

$$\begin{aligned}R_{r_1r_2}^{lr_1+r_2-l}=&\frac{1}{(1-\lambda_1\lambda_2)^{r_1+r_2}}\sum_{l_1+l_2=l}\binom{r_1}{l_1}\binom{r_2}{l_2}\\&\times(1+\lambda_1)(1-\lambda_2)^{r_1-l_1}(1-\lambda_1)(1+\lambda_2)^{l_2}(\lambda_1-\lambda_2)^{r_2-l_2}(\lambda_2-\lambda_1)^{l_1}.\end{aligned}\tag{36}$$

As pointed out in [15], due to the non-trivial limit of comultiplication of the generators, this limit of R gives a infinite dimensional representation of braid group. From the calculation, the unitarity of R is obvious

$$\sum_{\mu}R_{\mu r_1+r_2-\mu}^{r_1'r_2'}(\lambda_2,\lambda_1)R_{r_1r_2}^{\mu r_1+r_2-\mu}(\lambda_1,\lambda_2)=\delta_{r_1r_1'}\delta_{r_2r_2'}.\tag{37}$$

In principle the hexagonal equation for R can be shown from this explicit expression, but the calculation is very complicated and we do not want to do it here.

4. Clebsch–Gordan coefficients

The Clebsch–Gordan coefficients are very important in studying the decomposition of two irreducible representations. It has been shown that the tensor product of two semiperiodic representations of $SU_q(2)$ is completely reducible and can be decomposed into the direct sum of semiperiodic representations [15, 16]. The decomposed rules are given by

$$(\lambda_1, y_1)\otimes(\lambda_2, y_2) = \bigoplus_{l=0}^{N-1} (q^{2l}\lambda_1\lambda_2, y_1+\lambda_1^N y_2). \tag{38}$$

This means that

$$e_r(\xi(l)) = \sum_{r_1 r_2} K_r^{r_1 r_2}(\xi_1, \xi_2, \xi(l)) e_{r_1}(\xi_1)\otimes e_{r_2}(\xi_2) \tag{39}$$

where $K^l s$ are the Clebsch-Gordan coefficients for the semiperiodic representations. In order to find these coefficients, we need to use the comultiplication of F. First, considering the highest weight vector $e_0(\xi)$ of the semiperiodic representation $\xi(l) = (q^{2l}\lambda_1\lambda_2, y_{12})$

$$\begin{aligned}\Delta(K)e_0(\xi(l)) &= \lambda_l e_0(\xi(l))\\ &= \sum K_0^{r_1 r_2} q^{2(r_1+r_2)}\lambda_1\lambda_2 e_{r_1}(\xi_1)\otimes e_{r_2}(\xi_2)\end{aligned} \tag{40}$$

which gives that

$$r_1 + r_2 = l \bmod N. \tag{41}$$

Thus, the highest weight vector of $\xi(l)$ can be written as

$$e_0(\xi(l)) = \sum_{r_1+r_2=l} K_0^{r_1 r_2} e_{r_1}\otimes e_{r_2} + \sum_{r_1+r_2=l+N} K_0^{r_1 r_2} e_{r_1}\otimes e_{r_2}. \tag{42}$$

The operator $\Delta(E)$ acting on $e_0(\xi(l))$ give the following

$$\begin{aligned}&\left(\sum_{r_1+r_2=l} + \sum_{r_1+r_2=N+l}\right)\{K_0^{r_1 r_2}([r_1](\lambda_1 q^{r_1-1})e_{r_1-1}\otimes e_{r_2}\\ &\qquad +[r_2](1+\lambda_2 q^{r_2-1})\lambda_1 q^{2r_1} e_{r_1}\otimes e_{r_2-1})\} = 0.\end{aligned} \tag{43}$$

Because of the linear independence of $e_{r_1}\otimes e_{r_2}$, one can find the recursion relation of $K^l s$

$$\begin{aligned}&K_0^{r_1+1 r_2}[r_1+1](1+\lambda_1 q^{r_1}) + K_0^{r_1 r_2+1}[r_2+1](1+\lambda_2 q^{r_2})\lambda_1 q^{2r_1} = 0\\ &(r_1+r_2+1=l) \quad \text{or} \quad (r_1+r_2+1=N+l)\\ &K_0^{Nl} = K_0^{lN} = 0.\end{aligned} \tag{44}$$

It is easy to find

$$K_0^{rl-r} = \frac{1}{[l]!}(-1)^r \lambda_1^r q^{r(r-1)}(\lambda_1)_{l,r}(\lambda_2)_{l,l-r} \qquad (0 \le r \le l) \tag{45}$$

and

$$K_0^{r+l, N-r} = 0 \qquad (0 \le r \le N-l).$$

In fact, equation (44) was given in [15]. Hence equations (41), (44) and (45) give the highest weight vector. Consequently, the vectors with arbitrary weight can be obtained from the comultiplication $\Delta(F^n)$ acting on the highest weight vector $e_0(\xi)$. A useful formula is

$$\Delta(F^n) = \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix} F^j K^{n-j} \otimes F^{n-j} \tag{46}$$

which acting on e_0 gives

$$\begin{aligned} e_n(\xi(l)) &= \frac{1}{(\lambda_l)_{0,n}} \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix} \sum_{r=0}^{l} K_0^{rl-r} \lambda_1^{n-j} q^{2r(n-j)} \\ &\quad \times (\lambda_1)_{r,r+j} (\lambda_2)_{l-r,l-r+n-j} e_{r+j} \otimes e_{l-r+n-j} \\ &= \frac{1}{(\lambda_l)_{0,n}} \sum_{m=0}^{l+n} e_m \otimes e_{l+n-m} \left\{ \sum \begin{bmatrix} n \\ m-r \end{bmatrix} K_0^{rl-r} \right. \\ &\quad \left. \times (\lambda_1)^{n+r-m} q^{2r(n+r-m)} (\lambda_1)_{r,m} (\lambda_2)_{l-r,l+n-m} \right\} \end{aligned} \tag{47}$$

where $\lambda_l = q^{2l}\lambda_1\lambda_2$ and second summation over r must keep r, $l-r$, $m-r$ and $n-m+r$ be positive or zero. Introducing a quantity

$$F(j, m, n) = \sum \begin{bmatrix} n \\ m-r \end{bmatrix} K_0^{rl-r} (\lambda_1)^{n+r-m} q^{2r(n+r-m)} (\lambda_1)_{r,m} (\lambda_2)_{l-r,l+n-m} \tag{48}$$

and making use of the relation $e_{N+i} = ke_i$, one can find the explicit expression as

$$K_n^{m,l+n-m} = F(l, m, n) \qquad \begin{cases} l+n<N \\ l+n \geqslant N \quad \text{and} \quad l+n-N<m<N \end{cases} \tag{49}$$

and

$$\begin{aligned} K_n^{m,l+n-m-N} &= kF(l, m, n) \qquad (l+n \geqslant N, 0 \leqslant m \leqslant l+n-N) \\ K_n^{m-N,l+n-m} &= kF(l, m, n), (l+n \geqslant N, N \leqslant m \leqslant l+n). \end{aligned} \tag{50}$$

It is worthy to note that although equations (49) and (50) have the similar form, indeed, the functions $F(l, m, n)$ give the different expressions for parameters l, m and n at different areas.

5. Discussion

In quantum group, the intertwiner and Clebsch–Gordan coefficients have the following relation

$$R(\Pi_1, \Pi_2) K(\Pi_1, \Pi_2, \Pi_3) = \phi(\Pi_1, \Pi_2, \Pi_3) K(\Pi_1, \Pi_2, \Pi_3) \tag{51}$$

where

$$\Pi_i : SU_q(2) \to End(V_i) \tag{52}$$

and ϕ is a scale function. For V_i being the irreducible $SU_q(2)$-moduule with spin j_i, the scale ϕ is

$$\phi(j_1, j_2, j_3) = (-1)^{j_1+j_2-j_3} q^{c(j_3)-c(j_1)-c(j_2)} \tag{53}$$

where $c(j) = j(j+1)$ is the classical casmir in V_j. For semiperiodic representation in the case $q^3 = 1$. Gomez *et al* have shown that the scale ϕ is unit and presume that this is true for arbitrary $N(q^N = 1)$. In principle, one can use the expressions of R (33) and K (49) and (50) to prove the conjecture. But it is very complicated and we do not give the proof of it.

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